9.1. Finiteness of class numbers

$K/\mathbb{Q}$ finite, $[K:\mathbb{Q}] = n$, $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$

Thm: $0 \neq I \subseteq K$ fract. ideal

1) $\exists\, x \in I \setminus \{0\}$, s.t.

$$|N_{K/\mathbb{Q}}(x)| \leq \left(\frac{4}{\pi}\right)^{r_2} \cdot \frac{n!}{n^n} \cdot \sqrt{|\Delta_K|} \cdot N(I)$$

2) $[I] = [\mathfrak{a}]$ in Cl_K for some $\mathfrak{a} \subseteq \mathcal{O}_K$, s.t.

$$N(\mathfrak{a}) \leq \left(\frac{4}{\pi}\right)^{r_2} \frac{n!}{n^n} \sqrt{|\Delta_K|}$$

In part, Cl_K is finite

Proof: 1) $\Rightarrow$ 2): Apply 1) to I^{-1}

$\Rightarrow x \in I^{-1}$, s.t.

$$|N_{K/\mathbb{Q}}(x)| \cdot N(I) \leq \left(\frac{4}{\pi}\right)^{r_2} \cdot \frac{n!}{n^n} \sqrt{|\Delta_K|}$$

By constr. $\underbrace{x \cdot I}_{= \mathfrak{a}} \subseteq I \cdot I^{-1} = \mathcal{O}_K$ $\square$

For 1): Fix $\sigma_1, \dots, \sigma_{r_1} : K \hookrightarrow \mathbb{R}$

$$\sigma_{r_1+1}, \dots \dots, \sigma_{r_1+2r_2} : K \to \mathbb{C}$$

$$\overline{\sigma_{r_1+2j}} = \sigma_{r_1+2j-1} \quad , j = 1, \dots, r_2$$

For $x \in K$

$$\Rightarrow \quad |N_{K/\mathbb{Q}}(x)| = \prod_{i=1}^{r_1} |\sigma_i(x)| \cdot \prod_{j=1}^{r_2} |\sigma_{r_2+2j}(x)|^2$$

$$\leq \frac{1}{n^n} \left(\sum_{i=1}^{r_1} |\sigma_i(x)| + 2 \sum_{j=1}^{r_2} |\sigma_{r_1+2j}(x)| \right)^n$$

inequality of geom. arithm. mean

Set $B_t := \{ (y_1, \dots, y_{r_1}, z_1, \dots, z_{r_2}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \mid \sum_{i=1}^{r_1} |y_i| + 2 \sum_{j=1}^{r_2} |z_j| \leq t \}$

$r_1 = 2, r_2 = 0$

$t \geq 0$

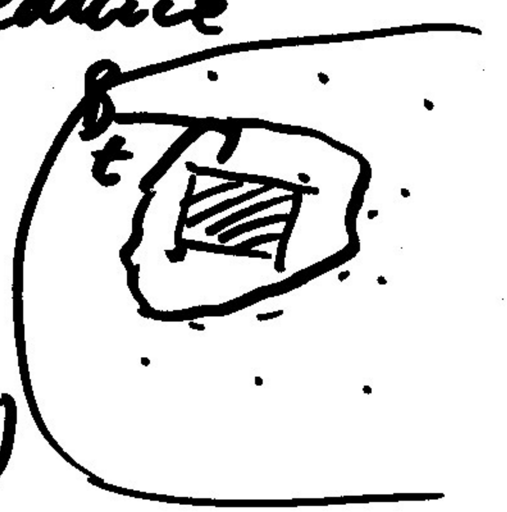
Want to find $x \in \iota(I) \cap B_t$ nonzero,

with $\iota : K \hookrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \simeq \mathbb{R}^n$

$$x \mapsto (\sigma_1(x), \dots, \sigma_{r_1}(x), \sigma_{r_1+2}(x), \dots, \sigma_{r_1+2r_2}(x))$$

$$(x_1, \dots, x_{r_1}, z_1, \dots, z_{r_2}) \mapsto (x_1, \dots, x_{r_1}, \mathrm{Re}(z_1), \mathrm{Im}(z_1), \dots\dots)$$

Lemma: $\Lambda := \iota(I) \subseteq \mathbb{R}^n$ is a lattice (i.e. Λ is free of $\mathbb{Z}$-rk n, and $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda \simeq \mathbb{R}^n$)



and $\mathrm{vol}(\mathbb{R}^n/\Lambda) = \frac{1}{2^{r_2}} \sqrt{|\Delta_K|} \cdot N(I)$

Here, $\mathrm{vol}(\mathbb{R}^n/\Lambda) :=$

$$\underbrace{\mu}_{\text{Lebesgue measure}}\left(\sum_{i=1}^{n} [0,1] \cdot \iota_i \right), \text{ with } \iota_1, \dots, \iota_n \in \Lambda \text{ are } \mathbb{Z}\text{-basis}$$

Equiv,

$$\mathrm{vol}(\mathbb{R}^n/\Lambda) = |\det(x_1, \dots, x_n)|$$

$$\left\{ \sum_{i=1}^n r_i x_i \mid r_i \in [0,1] \right\}$$

Proof: $x(I)$ lattice as $I \otimes_{\mathbb{Z}} \mathbb{Q} = K$

Pick non-zero fract. ideals $J \subseteq I$, $J \subseteq \mathcal{O}_K$

$$\Rightarrow \mathrm{vol}(\mathbb{R}^n/x(I)) = \underbrace{[I:J]}_{N(I)\cdot N(J)}{}^{-1} \underbrace{\mathrm{vol}(\mathbb{R}^n/x(J))}_{\mathrm{vol}(\mathbb{R}^n/x(\mathcal{O}_K))\cdot N(J)}$$

$$= N(I)\cdot \mathrm{vol}(\mathbb{R}^n/x(\mathcal{O}_K))$$

$\Rightarrow$ wlog $I = \mathcal{O}_K$

Recall $\Delta_K = \det(\sigma_i(\alpha_j))_{i,j}^2$.

$\alpha_1, \dots, \alpha_n$ $\mathbb{Z}$-basis of $\mathcal{O}_K$

$$(\sigma_i(\alpha_j))_{i,j} = \begin{pmatrix} \mathrm{Id}_{r_1} & & 0 \\ & \square & \\ 0 & & \ddots \\ & & & \square \end{pmatrix} (\tau(\alpha_1), \dots, \tau(\alpha_n))$$

r_2-times 2×2-matrix $\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$ Re σ_{r_1+2j}, Im

$$\Rightarrow \det(\sigma_i(\alpha_j))_{i,j} = (-2i)^{r_2} \cdot \det(\tau(\alpha_1), \dots, \tau(\alpha_n))$$

$$\Rightarrow |\Delta_K| = 4^{r_2} \cdot \mathrm{vol}(\mathbb{R}^n / \tau(\mathcal{O}_K))^2$$

$$\Rightarrow \mathrm{vol}(\mathbb{R}^n / \tau(\mathcal{O}_K)) = \frac{1}{2^{r_2}} \cdot \sqrt{|\Delta_K|} \qquad \square$$

Lemma (Minkowski): $\Lambda \subseteq \mathbb{R}^n$ lattice,

⚠ $X \subseteq \mathbb{R}^n$ central sym, convex,

(i.e. $x \in X \Rightarrow -x \in X$, $0 \in X$

$x, y \in X, \lambda \in [0,1] \Rightarrow \lambda x + (1-\lambda) y \in X$)

s.t. $\mu(X) > 2^n \mathrm{vol}(\mathbb{R}^n/\Lambda)$.

$\Rightarrow X \cap \Lambda \neq \{0\}$

Proof: $P_e = \sum_{i=1}^{n} [0,2] \cdot \lambda_i$, $\lambda_1, \dots, \lambda_n$ $\mathbb{Z}$-basis of Λ

$\Rightarrow \mu(P_e) = 2^n \cdot \mathrm{vol}(\mathbb{R}^n/\Lambda)$

$$\mu(X) = \sum_{\lambda \in 2\Lambda} \mu((\lambda + P_e) \cap X)$$

$$= \sum_{\lambda \in 2\Lambda} \mu((-\lambda + X) \cap P_e)$$

The sets $(\lambda + X) \cap P_e$, $\lambda \in 2\Lambda$

cannot be disjoint, as otherwise

$$\mu(X) = \sum_{\lambda \in 2\Lambda} \mu((\lambda + X) \cap P_e) = \mu\Big(\bigcup_{\lambda \in 2\Lambda} (\lambda + X) \cap P_0\Big)$$

$$\leq \mu(P_e) \quad \lightning$$

$\Rightarrow \exists\, \lambda_1, \lambda_2 \in 2\Lambda$ & $x, y \in X$, s.t.

$\lambda_1 + x = \lambda_2 + y$ & $\lambda_1 \neq \lambda_2$

$$\Rightarrow \underset{\underset{\Lambda}{\uparrow}}{\frac{\lambda_1 - \lambda_2}{2}} = \underset{\underset{X}{\uparrow}}{\frac{y - x}{2}} \in X \cap \Lambda \setminus \{0\}$$

$\leftarrow$ as X centr. sym. convex. $\bullet$

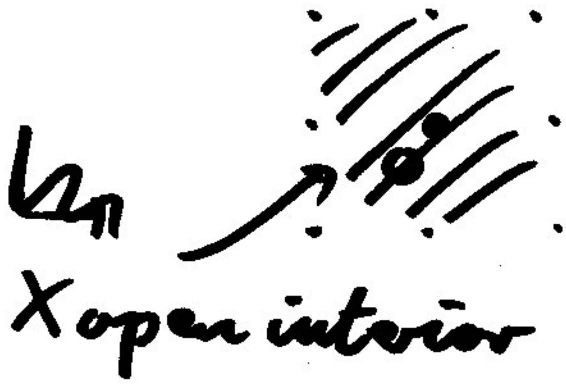
If X is compact, centr. sym, convex,

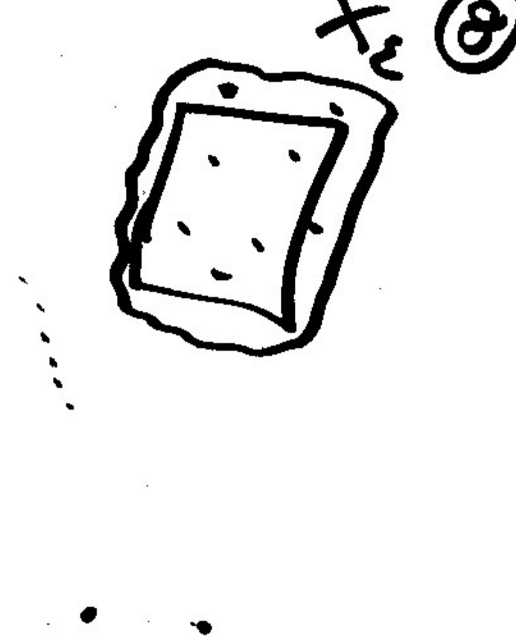
$\mu(X) \geq 2^n \cdot \mathrm{vol}(\mathbb{R}^n/\Lambda)$

$\Rightarrow X \cap \Lambda \neq \{0\}$

(apply le to $\underset{X_\varepsilon}{\underset{\parallel}{(1+\varepsilon) \cdot X}}$, $\varepsilon > 0$

& $X_\varepsilon \cap L$ is finite)





Want to ~~ap ap~~ apply this to

B_t, $t \gg 0$ (B_t centr sym, convex)

La: $\mu(B_t) = 2^{r_1} \cdot \left(\frac{\pi}{2}\right)^{r_2} \cdot \frac{t^n}{n!}$

Proof: Set $V(r_1, r_2, t) = \mu(B_t)$, $B_t \subseteq \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$

$n := r_1 + 2r_2$

$V(1,0,t) = 2t$ ✓

$V(0,1,t) = \pi (t/2)^2 = \left(\frac{\pi}{2}\right) \cdot \frac{t^2}{2!}$ ✓

Assume $V(r_1, r_2, t) = 2^{r_1} \left(\frac{\pi}{2}\right)^{r_2} \cdot \frac{t^n}{n!}$

$\Rightarrow$ 1) $V(r_1+1, r_2, t)$

$$= \int_{-t}^{t} V(r_1, r_2, t-|y|)\, dy$$

$$= 2\int_{0}^{t} 2^{r_1} \cdot \left(\frac{\pi}{2}\right)^{r_2} \frac{(t-y)^n}{n!}\, dy$$

$$= 2^{r_1+1} \cdot \left(\frac{\pi}{2}\right)^{r_2} \left[-\frac{(t-y)^{n+1}}{(n+1)!} \right]_0^t$$

$$= 2^{r_1+1} \left(\frac{\pi}{2}\right)^{r_2} \cdot \frac{t^{n+1}}{(n+1)!}$$

2) $V(r_1, r_2+1, t)$

$$= \int_{|z| \le \frac{t}{2}} V(r_1, r_2, t-2\cdot|z|)\, dz$$

$$= \int_0^{t/2} \int_0^{2\pi} V(r_1, r_2, t-2\varrho)\, \varrho \, d\theta \, d\varrho$$

$$= \ldots = 2^{r_1} \cdot \left(\frac{\pi}{2}\right)^{r_2+1} \cdot \frac{t^{n+2}}{(n+2)!}$$

$\square$

Let's collect what we have

If $\mu(B_t) \geq 2^n \cdot \mathrm{vol}(\mathbb{R}^n / \pi(I))$,

$\Rightarrow \pi(I) \cap B_t \neq \{0\}$

{Equiv., if $2^{r_1} \cdot \left(\frac{\pi}{2}\right)^{r_2} \cdot \frac{t^n}{n!} \geq 2^n \cdot \frac{1}{2^{r_2}} \cdot \sqrt{|\Delta_K|} \cdot N(I)$

$\Leftrightarrow t^n \geq \left(\frac{4}{\pi}\right)^{r_2} \cdot n! \cdot \sqrt{|\Delta_K|} \cdot N(I)$

& If $x \in K$, s.t. $\pi(x) \in B_t$,

$\Rightarrow |N_{K/\mathbb{Q}}(x)| \leq \frac{1}{n^n} \cdot t^n$

Choose $t^n = \left(\frac{4}{\pi}\right)^{r_2} \cdot n! \sqrt{|\Delta_K|} \cdot N(I)$

$\Rightarrow$ Thm proven ▪

Corollary: $\sqrt{|\Delta_K|} \geq a_n := \left(\frac{\pi}{4}\right)^{n/2} \cdot \frac{n^n}{n!}$

with $1 < a_2 < a_3 < \ldots$

In part, $|\Delta_K| > 1$ if $K \neq \mathbb{Q}$

Proof: Thm 1) for $I = \mathcal{O}_K$

$$\Rightarrow \exists\, \alpha \in \mathcal{O}_K \setminus \{0\}, \text{ s.t.}$$

$$1 \underset{\substack{\uparrow \\ \alpha \in \mathcal{O}_K \setminus \{0\}}}{\leq} |N_{K/\mathbb{Q}}(\alpha)| \leq \left(\frac{4}{\pi}\right)^{r_2} \cdot \frac{n!}{n^n} \sqrt{|\Delta_K|}$$

$$\Rightarrow \sqrt{|\Delta_K|} \geq \frac{n^n}{n!} \cdot \left(\frac{\pi}{4}\right)^{r_2} \underset{r_2 \leq \frac{n}{2}}{\geq} \frac{n^n}{n!} \left(\frac{\pi}{4}\right)^{n/2}$$

Finally,

$$\frac{a_{n+1}}{a_n} = \sqrt{\frac{\pi}{4}}\, \frac{(n+1)^{n+1}}{(n+1)!}\, \frac{n!}{n^n} =$$

$$\sqrt{\frac{\pi}{4}} \left(1 + \frac{1}{n}\right)^{n}$$

$$> \sqrt{\frac{\pi}{4}} \left(1 + \frac{1}{2}\right)^2 > 1$$

$\square$

Thm (Hermite): For fixed integer Δ, there exist only fin. many $K/\mathbb{Q}$ finite with $\Delta_K = \Delta$

Proof: Prev. cor. $\Rightarrow$ r_1, r_2 are bdd in terms of Δ

$\Rightarrow$ Can assume Δ, r_1, r_2 are fixed for K

For such K construct $\alpha \in \mathcal{O}_K$ with $K = \mathbb{Q}(\alpha)$ & coeff. of min. poly of α are bdd in terms of Δ, r_1, r_2

$\mathbb{Z}[x]$

Thm: $0 \neq I \subseteq K$ frac. ideal.

$c_1, \dots, c_{r_1+r_2} > 0$

with $\displaystyle\prod_{i=1}^{r_1+r_2} c_i > \left(\frac{2}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|} \cdot N(I)$

$\Rightarrow$ ex. $\alpha \in I \setminus \{0\}$, s.t. $|\sigma_i(\alpha)| < c_i$

for $1 \leq i \leq r_1$ and

$|\sigma_{r_1+2j}(\alpha)|^2 < c_{r_1+j}$, $j = 1, \dots, r_2$

Proof: $X := \{(y_1, \dots, y_{r_1}, z_1, \dots, z_{r_2}) \in \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \mid$
$|y_i| < c_i,\ |z_j|^2 < c_{r_1+j}\ \forall i,j\}$

$\Rightarrow X$ centr. sym, convex

$$\mu(X) = 2^{r_1} \cdot \pi^{r_2} \cdot \prod_{i=1}^{r_1+r_2} c_i$$

$$> 2^{r_1} \cdot \pi^{r_2} \cdot \left(\frac{2}{\pi}\right)^{r_2} \cdot \sqrt{|\Delta_K|} \cdot N(I)$$

$$= 2^n \cdot \mathrm{vol}(\mathbb{R}^n / \iota(I))$$

$\Rightarrow$ apply Minkowski's lemma $\square$

Assume $r_1 > 0$

Choose $c_i > 0$, s.t. $c_1 > 1$, $c_i < 1$ for $i > 1$

and $\prod_{i=1}^{r_1+r_2} c_i > \left(\frac{2}{\pi}\right)^{r_2} \sqrt{|\Delta_K|} = \left(\frac{2}{\pi}\right)^{r_2} \sqrt{|\Delta|}$

Pick $\alpha \in \mathcal{O}_K \setminus \{0\}$ as thm

$\Rightarrow 1 \leq |N_{K/\mathbb{Q}}(\alpha)| = \prod_{i=1}^{r_1} |\sigma_i(\alpha)| \cdot \prod_{j=1}^{r_2} |\sigma_{r_1+2j}(\alpha)|^2$

$\Rightarrow |\sigma_1(\alpha)| > 1 > |\sigma_i(\alpha)|$, $i \neq 1$

In part, $\sigma_1(\alpha) \neq \sigma_i(\alpha)$ $\forall i \neq 1$

$\Rightarrow K = \mathbb{Q}(\alpha)$ & coeff of min. polynomial are bdd in terms of $|\sigma_i(\alpha)|$, i.e. by the c_i $\Rightarrow$ Done

Assume $r_1 = 0$

Consider for $c > 0$

$$X = \{ (z_1, \dots, z_{r_2}) \in \mathbb{C}^{r_2} \mid |\mathrm{Re}(z_1)| < \tfrac{1}{2},$$
$$|\mathrm{Im}(z_1)| < c,$$
$$|z_j|^2 < c_j := \tfrac{1}{2},$$
$$\forall\, 2 \le j \le r_2$$

$\Rightarrow$ X centr. sym., convex

Pick $c_1 \gg 0$, s.t.

$$\mu(X) > 2^n \cdot 2^{-r_2} \cdot \sqrt{|\Delta|}$$

$\Rightarrow$ $\exists\, 0 \neq \alpha \in X \cap r(\mathcal{O}_K)$

$\Rightarrow$ (as before) $|\sigma_1(\alpha)| > 1 > |\sigma_j(\alpha)|$ $\forall\, j \ge 2$, $j \le r_2$

As $|\mathrm{Re}(\sigma_1(\alpha))| < \tfrac{1}{2}$

$\Rightarrow$ $\mathrm{Im}(\sigma_1(\alpha)) \neq 0$

$\Rightarrow$ $\sigma_1(\alpha) \neq \sigma_2(\alpha)$

$\Rightarrow$ $K = \mathbb{Q}(\alpha)$ & (coeff. of min poly are bdd in terms of $c_1, \dots, c_{r_2}$ $\square$